Research Statement – January 2009

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1. Overview

The principal aim of my research is to relate representation theory to geometry or topology, and to use such relations to determine multiplicity formulas or to find equivalences in a context of Koszul or Langlands duality. The central method that I use is to compare both sides to a category that is defined by combinatorial means. Here are my main research topics:

- The critical level representation theory of affine Kac–Moody algebras in the framework of the local geometric Langlands conjectures.
- A Koszul duality for representations of quantum groups and reductive algebraic groups.
- The positivity of the coefficients of arbitrary Kazhdan–Lusztig polynomials.
- A combinatorial description of perverse sheaves on the semi-infinite flag manifold.
- The intersection cohomology of affine Schubert varieties with coefficients in positive characteristics.

2. Results

The guiding example of the sort of relation between representation theory and geometry that is central for my research is the correspondence between intersection cohomology sheaves on flag varieties and projective objects in a highest weight category of a Kac–Moody algebra (cf. [Fie1, Fie2, Fie4]). This correspondence led to a new proof of the affine Kazhdan–Lusztig conjectures, as well as to a Koszul duality in the Kac–Moody case.

In [Fie7] I constructed such a relation for the context of algebraic groups over a field of positive characteristics. More concretely, I found a relation between a certain category of sheaves of k-vector spaces on the affine flag variety and the category of projective representations of a quantum group, if $k = \mathbb{C}$, or of a Lie algebra over k, if $\operatorname{char} k > 0$. As a corollary one obtains that the Lusztig conjecture on multiplicities of simple rational representations of a reductive algebraic group holds in almost all characteristics ([Fie8]).

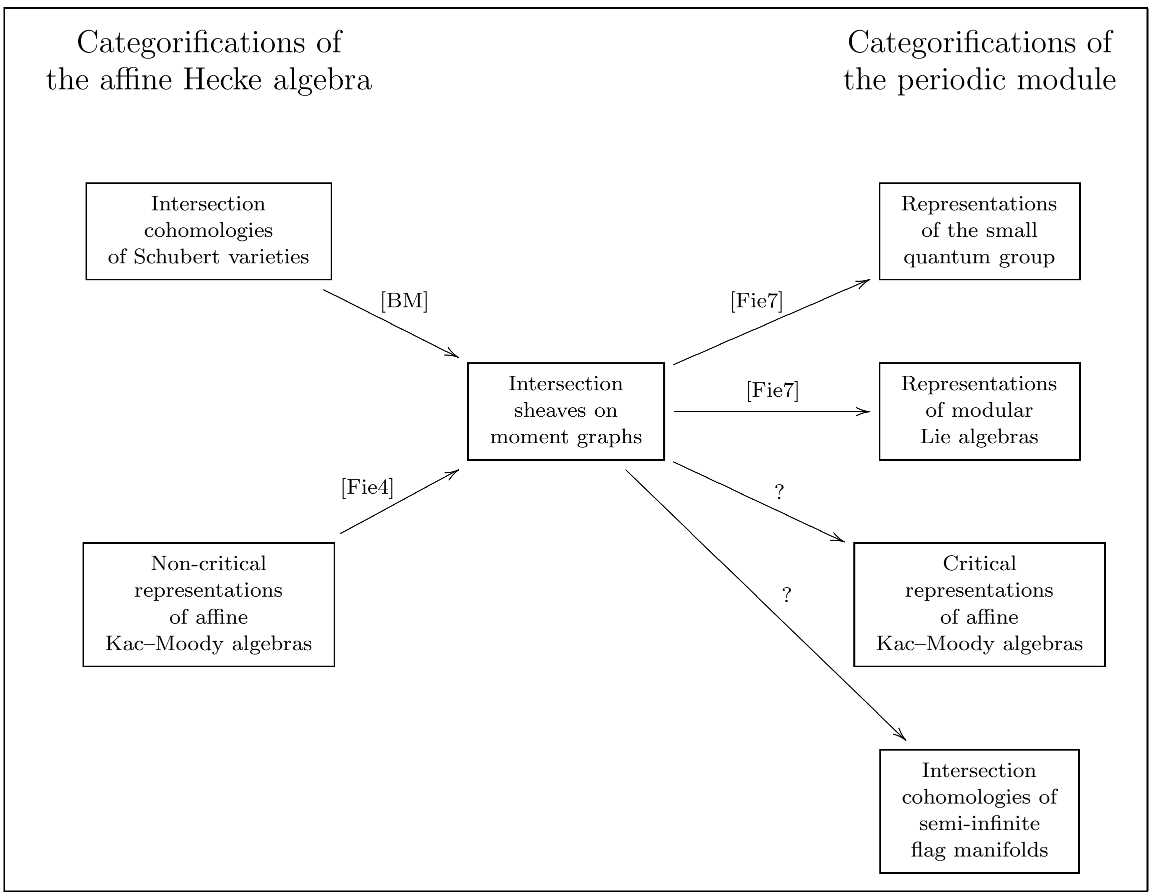
The result in [Fie7] is obtained by linking the representation theoretic and the geometric sides to a certain category of sheaves on an associated moment graph. A multiplicity conjecture for the occuring sheaves then implies Lusztig's conjecture. In [Fie6] I proved the multiplicity one case of this conjecture for all relevant fields, and hence the multiplicity one case of Lusztig's conjecture in full generality. Moreover, the above relation also yields an upper bound on the exceptional characteristics (cf. [Fie9]). Even though this bound is huge, it is the only one known so far.

The newest result of my research is obtained in a joint effort with Tomoyuki Arakawa (cf. [AF]). We studied critical level representations of an affine Kac–Moody algebra and proved the subgeneric cases of the Feigin–Frenkel conjecture. The above preprint is the first in an intended series of articles devoted to a program that aims at proving the Feigin–Frenkel conjecture in full generality. Moreover, we hope to answer a question of Lusztig on the anticipated relation to quantum group representations.

The paper [Fie3] explains the first steps in a program that aims to formulate and study the decomposition theorem for arbitrary Coxeter systems. In [Fie5] an equivalence between the category of intersection sheaves on Bruhat graphs and the category of Soergel's bimodules is constructed. The latter category is intimately connected to the new knot invariants of Khovanov.

3. Projects

The diagram above gives a broad overview on the main relations that are already established or are the goals of my current work.



In the following I want to give a few more details about the ideas underlying my research projects.

3.1. **Representations of affine Kac–Moody algebras in the critical level and the local geometric Langlands conjectures.** Linking representation theory to geometry is one of the cornerstones of all incarnations of the Langlands program. In particular, recent approaches towards the geometric Langlands conjectures try to relate the representation theory of affine Kac–Moody algebras to the geometry of affine Grassmannians ([BD]).

Edward Frenkel and Dennis Gaitsgory recently outlined an extensive research project on the local geometric Langlands conjectures in [FG]. Let G be a complex algebraic reductive group and denote by $G^\vee$ its Langlands dual. One of the aims of the Frenkel–Gaitsgory project is to associate to a $G^\vee$-local system σ on the infinitesimal punctured complex disc a certain category $\mathcal{C}_\sigma$ that is acted upon by the formal loop group $G((t))$.

The authors argue that there are at least two possible ways to realize $\mathcal{C}_\sigma$. The first is as a category of D-modules on the affine Grassmannian $\mathcal{G}\mathrm{r} = G((t))/G[[t]]$ that satisfy the Hecke eigenproperty with respect to σ. The second is as a category of representations with critical level of the associated affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ with central character related to σ. This led to a series of deep conjectures that connect the geometry of $\mathcal{G}\mathrm{r}$ to the critical representation theory of $\widehat{\mathfrak{g}}$.

My principal research project is motivated by the Frenkel–Gaitsgory program. One of its main goals is to define and to study the category of *restricted* representations for $\widehat{\mathfrak{g}}$ with critical level. This category corresponds to certain integral local systems σ and naturally comes up in the Frenkel–Gaitsgory approach. I believe that it is intimately connected to the representation theory of the corresponding small quantum group. This would yield multiplicity formulas for simple objects in the critical hyperplane and hence prove what is known as Feigin's conjecture.

3.2. **A Koszul duality for quantum groups and for modular Lie algebras.** There are at least two ways to obtain relations between representation theory and geometry. The first is using a Beilinson–Bernstein localization functor, which amounts to finding a suitable variety and to realize Lie algebras as differential operators.

My research focuses on a second, and very different approach in the spirit of the pioneering work of Andersen, Jantzen and Soergel. For this one has to find a combinatorially defined category that can be simultaneously linked to either geometry or representation theory.

The approaches above are Koszul dual in the following sense. Typically the Beilinson–Bernstein functor relates simple holonomic D-modules, or intersection cohomology complexes, to simple representations. The Jantzen–Soergel approach, however, relates intersection cohomology sheaves to projective representations. In addition, the underlying geometry is naturally associated to the algebraic structures with the Langlands dual datum.

The recent results of Bezrukavnikov and his coworkers (cf., for example, [ABG] for the quantum group case) on the localizations of modular and quantum group representations should in my opinion be considered of Beilinson–Bernstein type, while the paper [Fie7] is founded on the Andersen–Jantzen–Soergel philosophy. Hence one might hope that a better understanding of the interplay between both approaches yields a Koszul duality in the quantum and the modular cases.

3.3. **The positivity conjecture for arbitrary Coxeter groups (the combinatorial decomposition theorem).** The most useful theorem in geometric representation theory is the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber in [BBD]. It underlies almost all proofs of multiplicity conjectures of Kazhdan–Lusztig type. However, so far it can only be proven with coefficients in a field of characteristic zero.

One can rather easily translate the statement of the decomposition theorem to certain combinatorial categories ([Fie3]) that are associated to arbitrary Coxeter systems. An easy corollary of the statement would give the positivity conjecture for the corresponding Coxeter system. But so far all attempts towards a combinatorial proof of this theorem failed. One hopes that such a proof would also determine the set of exceptional primes (in the crystallographic cases), which would constitute the final step for a full proof of Lusztig's conjecture. The analogous results in the case of toric varieties by Karu in [Kar] certainly provide hope for finding such a translation.

3.4. **The semi-infinite flag manifold.** A very new geometric object that entered the world of geometric Langlands is the mysterious semi-infinite flag manifold. One of its main problems is that it is not of finite dimension, nor can it be approximated by finite dimensional manifolds. This makes the usual definition of perverse sheaves impossible (an ad-hoc substitute is constructed in [ABBGM]).

I believe that one can also define a *combinatorial* category as a substitute for the category of perverse sheaves on a semi-infinite flag manifold. The reason for this belief is contained in [Fie7]. Two specific combinatorial categories associated to an affine Weyl group are considered there: the first, denoted by $\mathcal{H}$, serves as a categorification of the affine Hecke algebra, while the second, denoted by $\mathcal{M}$, categorifies its periodic module. Then a functor $\Phi\colon \mathcal{H} \to \mathcal{M}$ is constructed.

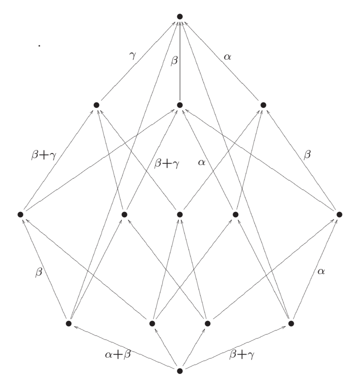
The category $\mathcal{H}$ encodes the structure of intersection cohomology complexes on the affine Grassmannian. There is some evidence that points to the fact that $\mathcal{M}$ is related to the geometry of the semi-infinite flag manifold, and that Φ is a natural restriction functor.

3.5. **The p-smooth locus of Schubert varieties.** In [Fie6] the smooth locus of self-dual moment graphs over arbitrary fields k is determined. If $k = \mathbb{C}$, this determines the rationally smooth locus of equivariantly formal varieties by results of Braden and MacPherson ([BM]). In a joint research project with Geordie Williamson I try to recover an analogue in positive characteristic.

4. Methods

Quite often duality statements between geometric and representation theoretic situations are given by parametrizing suitable standard objects on both sides by the *same* combinatorially defined set. In such a case one expects that it is even possible to relate the categorical structures, hence to categorify an already established bijection of sets. One might hope that the parameter set itself carries additional structure and hence gives rise to a "combinatorial" category which forms the sought-after link.

In Lie theory the parameter set very often is an orbit of an associated Weyl group (cf. the examples of category $\mathcal{O}$ or the Bruhat decomposition of flag varieties). The action of the Weyl group on these orbits then provides the additional structure. This is most conveniently encoded in a *moment graph*. A moment graph is a directed graph without cycles, whose edges are labelled by non-zero elements of a vector space over an arbitrary field.



A moment graph

One of the principal advantages of the combinatorial descriptions is their approachability by computer calculations. While it is often not feasible, or even impossible, to implement sample computations for infinite-dimensional geometric or algebraic structures, the computations in the combinatorial categories break down to certain finite arithmetics that hopefully can be implemented quite easily. Such sample computations might lead to a better understanding of the representation theory, or at least to new conjectures.

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